Advance Approach of Finite Dimensional Frame Theory and its Difference Application in Engineering

Satpal Panika & Dr. S. Biswas Kalinga University Naya Raipur

ABSTRACT

The study of *frames* in finite dimensional Hilbert spaces. *Frames*, as a whole, are a very broad area and our focus is clearing the basic concept and it's important in basic science as well as in various branches of applied engineering. Previously, frames were used in signal and image processing, non harmonic Fourier series, Data compression, sampling theory. Now a day's frame theory has even increasing application each and every area of mathematics also its different applications and uses in Physics, Computer Science.

Key words: Frames, Hilbert space, Redundant set, Orthogonality, Parseval sequence , Quadratic forms, eigenvalues

INTRODUCTION

Linear algebra is the study of vectors and linear functions i.e. in another words ,vectors are things you can add and linear functions are functions of vectors that respect vector addition. The goal of this study is to organize information about vector spaces in such a way that makes problems involving linear functions of many variables easy.

The purpose of this section is to review some fundamental concepts and results from linear algebra and matrix theory that will be needed in the main concept of Frame theory. Linear algebra concepts are as important as the computations. Here, it is considered vector spaces over the real numbers R or the complex number C.Many natural examples of vector spaces are collections of function under the usual operations of multiplication by scalars and addition of functions. A fundamental idea in the study of vector spaces is the practice of combining certain elements together to form others.

Hilbert Space Theory: The development of Hilbert space, and its subsequent popularity, was a result of both mathematical and physical necessity. The historical events and individuals responsible for the topics we will cover make up an interesting story.

Hilbert space gives a means by which one can consider functions as points belonging to an infinite dimensional space. The utility of this perspective can be found in our ability to generalize notions of orthogonality and length to collections of objects (i.e. functions) which do not naturally suggest the consideration of these properties. While Hilbert space eventually turned out to be the desired setting for quantum mechanical systems, its applications to physics were not the only motivation for its conception.

FRAME

Definition of frame:

The notion of bases in finite-dimensional spaces implies that the number of representative vectors is the same as the dimension of the space. When this number is larger, we can still have a representative set of vectors, except that the vectors are no longer linearly independent and the resulting set is then called a **frame**.

Frames are signal representation tools which are redundant. Redundant set of vectors $\Phi = \{\phi i\}_{i\in I}$ is called a frame while $\tilde{\Phi} = \{\tilde{\phi} i\}_{i\in I}$ is called a dual frame of $\Phi = \{\phi i\}_{i\in I}$.

R^n-Frames and Parseval's Frames

One of the important concepts in the study of vector spaces is the concept of a basis for the vector space, which allows every vector to be uniquely represented as a linear combination of the basis elements. However, the linear independence property for a basis is restrictive; sometimes it is impossible to find vectors which both fulfill the basis requirements and also satisfy external condition demanded by applied problems.

For such purposes, I need to look for more flexible types of spanning sets. Frames provide these alternatives. They not only have great variety for use in applications, but also have a rich theory from a pure analysis point of view.

An *R^n– frames* is a finite sequence of vectors $F = \{v_1,v_2,v_3, . . . ,v_k\}$ in R^n, with $k\geq n$, such that there exists an extension of F to a basis for R^k. A finite sequence of vectors $\{v_1,v_2,v_3, . . . ,v_k\}$ in R^n is called a *Parseval sequence* if it satisfies the Parseval's identity. Hence every Parseval sequence for R^n is necessarily an R^n-frame. And a Parseval frame $\{v_1,v_2,v_3, . . . ,v_k\}$ for R^n has an extension to an orthonormal basis of R^k.

In the Hilbert space, A sequence of vector $\{x_i\}\subset H$ where i = 1 to k, is called the parseval frame for H if for every $x \in H$

$$\| x \|^2 = \sum_{i=1}^{n} | < x ,e_i> |^2.$$

And the reconstruction formula is define by

$$x = \sum_{i=1}^{n} < x ,x_i >x_i .$$

Proposition: Let $\{ x_i\}_{i=1}^{k}$ be a Parseval frames for a Hilbert space H. Then it is an orthonormal basis if and only if each x_i is a unit vector.

Proof: The forward direction is obvious by the definition of orthonormal basis, so we need to show that $\{ x_i\}_{i=1}^{k}$ is an orthonormal basis when $\| x_i \| = 1$ for all i= 1,2 . . .,k. We know that it is an orthonormal set. Since we assumed $\{ x_i\}_{i=1}^{k}$ is a parseval frame for H, the set must span H , therefore must be an orthonormal basis for H.□

And if these finite vectors are Parseval frames for H then the dimension of Hilbert space is $\sum_{i=1}^{k}.\|x_i\|^2$. Let A be an operator on H. Then trace of A is defined by

$$tr(A) = \sum_{i=1}^{k} < Ax_i , x_i >.$$

Next section we are discuss the general Frames

General Frames

The general frames introduce the help of Parseval frames. A frames for a Hilbert space H is a sequence of vectors $\{x_i\}_{i=1}^{k} \subset H$ for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in H$,

$$A\|x\|^2 \leq \sum |<x, x_i>|^2 \leq B\|x\|^2.$$

A & B are called optimal frame bounds. If A=B, the frame is called a Tight Frame. If A=B=1, the frame is called a Parseval Frame. A uniform frame is a frame in which all the vectors have same norm.

Similar and Unitary Equivalent frames

There are several commonly used notion of equivalence among frames. In other words, there are frames which, although they are technically they are technically different, are considered to the same. Some natural notions of equivalent frames in R^n.

(i) Frames which contain the same vectors, but given in a different order. (ii) Frames which only differ in that some of the vectors x_i have been replaced with the additive inverse $-x_i$.

(ii) Frames which are rotations of each other, in the sense that a fixed rotation is applied to each vector of one frame to create the other frame.

There are more general notions of equivalence called the similarity of frames and the unitary equivalence of frames. Two frames $\{x_i\}_{i=1}^{k}$ and $\{y_i\}_{i=1}^{k}$ for Hilbert spaces H and K respectively are said to be *similar* if there exists an invertible operator $T: H \to K$ such that $Tx_i = y_i$ for $i = 1, 2, \ldots, k$. The frames are called *unitarily equivalent* if we require T to be a unitary operator from H to K. Hence similarity is an equivalence relation which is order dependent. For example, if $\{e_i\}_{i=1}^{k}$ is an orthonormal basis for H, then $\{0, e_1, e_2, \ldots, e_k\}$ and $\{e_1, 0, e_2, \ldots, e_k\}$ are two no similar frames, although they are the same set. But they are equivalence.

Frames in R^2

We have develops a geometric description of tight frames in the plane R^2.

Introduction of Diagram vectors

We begin by using polar coordinates to express any vector x in R^2 in the form $x = \begin{bmatrix} a\cos\theta \\ a\sin\theta \end{bmatrix}$, where θ is the angle the vector makes with the positive x- axis.A collection of vectors $\{x_i\}_{i=1}^{k}$,where $x_i = \begin{bmatrix} a_i\cos\theta_i \\ a_i\sin\theta_i \end{bmatrix}$,is a tight frame with frame bound A if and only if the frame operator S is equal to A time the identity operator.

If $x = \begin{bmatrix} a\cos\theta \\ a\sin\theta \end{bmatrix}$ is a vector in R^2, let $\tilde{x} = \begin{bmatrix} a^2 \cos 2\theta \\ a^2 \sin 2\theta \end{bmatrix}$ be this associated vector which we will call the diagram vector, because we will be drawing diagram in which we discuss the sum of each vectors. A collection $\{x_i\}_{i=1}^{k}$ of vector in R^2, where $k \geq 2$ is a tight frame for R^2 if the diagram vectors $\{\tilde{x}_i\}_{i=1}^{k}$ sum to zero in R^2.

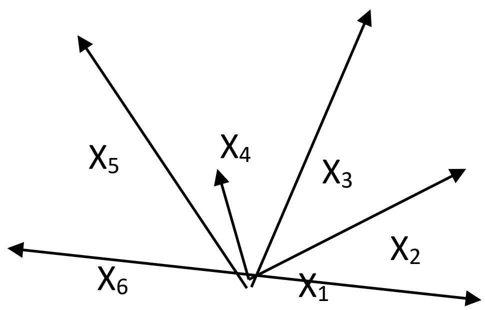

Fig. (i) Tight frame for R^2

The diagram of sum of vectors $\{\tilde{x}_i\}_{i=1}^{k}$ provides a visual representation of the tight frames in R^2, and provides very intuitive answer to questions we may ask about frames.

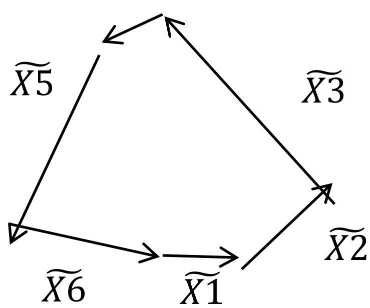

Fig (2) The diagram vector of position figure (1)

If we want to consider all the tight frames having three vectors, we can consider the possible diagrams of unit vectors which can sum to zero.

Similar and Equivalent frames

Some natural notions of equivalent frames in R^n.

(i) Frames which contain the same vectors, but given in a different order.

(ii) Frames which only differ in that some of the vectors x_i have been replaced with the additive inverse $-x_i$.

(iii) Frames which are rotations of each other, in the sense that a fixed rotation is applied to each vector of one frame to create the other frame.

There are more general notions of equivalence called the similarity of frames and the unitary equivalence of frames than the Equivalence based on the notions above. Nonetheless, all these talk about equivalence but each having different criteria.

APPLICATIONS

Finite Frames:

It is versatile methodology for any application which requires redundant, yet stable decompositions. For example, for the study or transmission of signals, but surprisingly also for more theoretically oriented questions.

Noise and Erasure Reduction

Noise and erasures are one of the most common problems signal transmissions have to face. The redundancy of frames is particularly suitable to reduce and compensate for such disturbances. Theoretical error considerations range from worst to average case scenarios.

Resilience against Perturbations

Perturbations of a signal are an additional problem faced by signal processing applications. Various results on the ability of frames to be resilient against perturbations are known. One class focuses on generally applicable frame perturbations results some even in the Banach space . Yet another topic are perturbations of specific frames such as Gabor frames, frames containing a Riesz basis , or frames for shift-invariant spaces . Finally, also extensions such as fusion frames are studied with respect to their behavior under perturbations.

Compressed Sensing

Since high dimensional signals are typically concentrated on lower dimensional subspaces, it is a natural assumption that the collected data can be represented by a sparse linear combination of an appropriately chosen frame. The novel methodology of Compressed Sensing, initially developed, utilizes this observation to show that such signals can be reconstructed from very few non-adaptive linear measurements by linear programming techniques. Finite frames thus play an essential role, both as sparsifying systems and in designing the measurement matrix.

Optimization problems

Engineers, economists, scientists, and mathematicians often need to find the maximum or minimum value of a quadratic form some specified set. Spectral theorem can be used to solve a few types of optimization problems as well. Spectral theorem also says that a real symmetric matrix can be brought to a diagonal form by an orthogonal matrix. In other words, a symmetric matrix is orthogonally diagonalizable.

A matrix A is said to be orthogonally diagonalizable if there exists an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Quadratic forms

A quadratic form on R^n is a function Q defined on R^n whose value at a vector x can be computed by the expression $Q(x) = x^TAx$, where A is a real symmetric matrix. It is the same as Q(x) = <Ax, x>. For example $A = \begin{matrix} 4 & 0 \\ 0 & 3 \end{matrix}$.. Quadratic form associated with A is $4x_1^2 + 3x_2^2$. Also from a given quadratic form, we can extract the symmetric matrix associated with it. Many Quadratic forms will have cross product terms, which we can deal with by a change of variable.

In simple words, if we diagonalize the matrix its quadratic form will have no cross-product terms. The accurate change of variable is $x = P y$ or $y = P^{-1} x$.

This works because $x^TA\,x = (P\,y)^TA\,(P\,y) = y^TP^TA\,P\,y = y^T(P^T A\,P)\,y = y^T D\,y$.

This change of variable is called as Principal Axes Theorem.

Optimization of Quadratic forms

We can use the above concepts to find a solution to the constrained optimization problems.

Consider a problem: Find the maximum and minimum values of $Q(x) = x^TAx$, A need not be a diagonal matrix, subject to the constrain $x^Tx = 1$.

Solution: The constraint is simply that x should be a unit vector in R^n. So, if we geometrically plot Q(x), we have to find maximum and minimum values of the intersection of Q (x) and a unit cylinder. Define

$m = \min \{ x^TAx : ||x|| = 1\}$ and $M = \max \{ x^TAx : ||x|| = 1\}$ (1)

The following theorem will give us our solution

Theorem :Let A be a real symmetric matrix. M and m as defined in (1) are nothing but the largest and smallest eigenvalues of A, respectively.

Proof: After orthogonally diagonalizing as PDP^{-1}, we get

$$x^TAx = y^TDy, \quad \text{where } x = Py$$

So,

$$||x|| = ||P\,y|| = ||\,y\,|| \text{ for all } y$$

Since $P^TP = I$ and $||\,P\,y\,||^2 = (P\,y)^T(P\,y) = y^T(P^TP)\,y = y^Ty = ||\,y\,||^2$

Now suppose A is a 3x3 matrix with eigenvalues $a \geq b \geq c$.

The eigenvector columns of P = [u_1 u_2 u_3]

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Let $y \in R^3$ with coordinates y_1, y_2, y_3.

$$ay_1^2 = ay_1^2$$
$$by_2^2 \leq ay_2^2$$
$$cy_3^2 \leq ay_3^2$$

Adding, we get $y^TDy \leq ay_1^2 + ay_2^2 + ay_3^2$

$$= a(\| y \|^2) = a$$

Thus, $M \leq a$. But $y^TD\ y = a$ when $y = (1, 0, 0)$. So $M = a$. Similarly we get $m = c$. Proof is completed.

CONCLUSION

we intended this study as a basic introduction to frames, geared primarily toward engineering students and those without extensive mathematical training. Frames are here to stay; asHilbert bases before them, they are becoming a standard tool in the signal processing toolbox, spurred by a host of recent applications requiring some level of redundancy. We hope this study will be of help when deciding whether frames are the right tool for the application in engineering.

References :

[1] Edwards, C.H and David E. Penney, *Differential Equations and Boundary Value Problems*, Pearson Education,Inc.,Third edition (2004)
[2] Herstein, I. N., *Topics in algebra*, Blaisdell Publishing Company, New-York, Toronto, London
[3] Han, D., Keri Kornelson, David Larson and Eric Weber, *Frames for Undergraduates* (2007) American Mathematical society
[4] Kreyszig, E., *Introductory functional Analysis with Applications*, John Wiley and sons, New York
[5] Lay, David C., *Linear Algebra & its Applications*, Dorling Kindersley (India) Pvt. Ltd. 2007
[6] Simmons, G.F., Introduction to topology and modern analysis, McGraw-Hill Book Co. Inc., International student edition (1963)
[7] Zhan, Xingzhi. , Paper titled *'Extremal eigenvalues of real symmetric matrices with entries in an interval'*, East China Normal University (2003)

[8]Peter G,Casaza,Gitta Kutyniok,and Friedrich Philipp Book titled " Introduction to Finite Frame Theory"

[9]Jelena Kovacevic and Amina Chebira paper titled " An Introduction to Frames" Fundamental and Trends in singnal Processing Vol.2 (2008)

Mr. Satpal Panika, **Kalinga University Naya Raipur**

Satpalpanika2013@gmail.com

Mobile No. 9826150937

Dr. S. Biswas,Kalinga University Naya Raipur

shubhb@yahoo.co.in

Mobile No.- 7587143432,